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DERIVATION OF STRUCTURAL DESIGN SENSITIVITIES FROM VIBRATION TEST DATA

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Structural sensitivities are often required in dynamic analyses of engineering structures, such as system identification and control, structural modification and optimization. Many different methods have been developed in the last three decades for the efficient computation of such sensitivities. Though these existing methods have proven to be very useful tools to structural analysts, they are restricted to those cases where accurate analytical or finite element models are available. In many practical applications where sensitivities are needed in the solution of troubleshooting problems only limited measured data are available; existing methods being inapplicable. In this paper, a new and effective method has been developed to derive structural design sensitivities, both frequency response function sensitivities and eigenvalue and eigenvector sensitivities, from limited vibration test data. Design sensitivities calculated directly from measured data are more accurate than those calculated from analytical or finite element models since structural modelling errors are inevitable due to the complexity of almost all engineering structures. The relationship between frequency response function sensitivities and eigenvalue and eigenvector sensitivities has been established. This relationship provides the theoretical basis for the experimental determination of eigenvalue and eigenvector sensitivities. The effect of residue (contribution of out-of-range modes) upon analysis accuracy has been examined. Detailed numerical assessment based on a turbine bladed disk model as well as experimental investigation of a beam structure have been conducted and the results have demonstrated the practicality of the proposed method.

1. INTRODUCTION

Sensitivity analysis, the study of changes in system characteristics with respect to parameter variation, is being used in a variety of engineering disciplines ranging from automatic control theory to the analysis of large-scale physiological systems [1]. Early applications of sensitivity analyses were mainly in the analysis of control systems—for example, the effect of system parameter changes on system stability and controllability [2]. Research interests in optimal control and automated structural optimization led to the development of gradient-based mathematical programming methods [3] in which sensitivities are used to find the search direction towards optimum solutions. More recently, research effort has been more focused on the applications of sensitivity analysis to systematic structural optimization as a practical tool for design engineers when large and complicated engineering structures are considered [4]. Because design optimizations of large structural systems require excessively long computational time and hence enormous computational cost due to calculation of structural sensitivities, much interest has recently been directed towards the development of efficient computational procedures. Also, sensitivity analyses have been applied to structural modification analysis, system identification and assessment

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of design trends and has become a versatile design tool in its own right [1]. Three basic types of structural sensitivity frequently used in engineering practice are the sensitivities of structural response (both static and dynamic), sensitivities of eigenvalues and eigenvectors, and sensitivities of optimum structural design with respect to problem parameters [1]. This paper is limited to structural sensitivities of dynamic response under harmonic loading, which are equivalent to sensitivities of frequency response functions, and those of eigenvalues and eigenvectors. Specifically, the paper focuses on the experimental determination of such sensitivities since no analytical model is assumed to be available, but some limited measured data are available.

There have been a number of different methods described in the open literature to calculate sensitivites of structural responses from analytical or finite element models. Gill *et al*. [5] developed an algorithm based on the concept of finite differences to compute response sensitivities with optimum finite difference step size. An effective analytical method was proposed by Haug and Arora [6] to compute design sensitivities of elastic mechanical systems. For second order derivatives of structural response which are often needed in approximate engineering analysis and design, an efficient method was proposed by Haftka [7] which requires all the first order derivatives to be available. In the case of dynamic loading, a sensitivity analysis method was proposed by Hsieh and Arora [8] which uses the adjoint variable approach by defining a vector of adjoint variables which satisfies a given differential equation. For the case of dynamic response to harmonic excitation which has special significance in structural dynamic analysis, the concept of receptance sensitivity and method for its calculation was discussed by Yoshimura [9]. The derivation of such receptance sensitivity was further improved by Lin and Ewins [10] in its application to the practice of structural finite element model improvement.

Alongside these studies, extensive research effort has also been devoted towards the development and application of eigenvalue and eigenvector sensitivity analysis. Jahn [11] derived complete formulae for first-order eigenvalue and eigenvector sensitivities for standard eigenvalue problems and applied them to improve an approximate set of eigenvalues and eigenvectors. The theory was later extended to the case of generalized symmetric eigenvalue problems by considering changes of physical parameters in the mass and stiffness matrices by Fox and Kapoor [12]. This method, later named as the "modal method", requires all the modes of the system to be available. This complete eigensolution is sometimes computationally expensive, especially when systems with large dimensions are considered. To avoid such difficulty, Nelson [13] developed an effective method to calculate eigenderivatives of a specific mode by just using the modal parameters of that mode. However, a matrix inverse of system dimension (in fact, of dimension $N-1$ where N is the dimension of the system) is required for each mode in order to solve the linear algebraic equations involved. In order to improve computational efficiency, an improved modal method, which aims to derive the required eigenderivatives approximately by using the calculated lower modes and the known flexibility matrix, was proposed by Lim *et al*. [14]. Recently, an improved method for the derivation of eigenderivatives has been proposed by Lin *et al*. [15] and has been successfully applied to the case of analytical model improvement. For eigenvalue and eigenvector sensitivities of repeated modes and for higher order sensitivity derivatives, discussions can be found in references [16, 17].

The existing methods which have been mentioned have proven to be very useful tools for structural analysts. However, they are restricted to those cases where accurate analytical or finite element models are available. In many practical applications, when a detailed analytical model of a structure under consideration is not available, but sensitivities are needed to solve troubleshooting problems, these existing methods become inapplicable. It is necessary therefore to develop alternative methods to calculate the required sensitivities using vibration test data. In this paper, a new and effective method is developed to calculate frequency response function sensitivities, and eigenvalue and eigenvector sensitivities from limited measured data. These sensitivities, derived experimentally, are more accurate than those from analytical models since structural modeling errors are inevitable in the analytical models due to the complexity of almost all engineering structures. The relationship between frequency response function sensitivities and eigenvalue and eigenvector sensitivities is established. This relationship provides the theoretical basis for the experimental determination of eigenvalue and eigenvector sensitivites. the effect of residue (contribution of out-of-range modes) upon analysis accuracy is examined. Detailed numerical assessment based on a turbine bladed disk model as well as experimental investigation of a beam structure are carried out and the results demonstrate the practicality of the proposed method.

2. THEORETICAL DEVELOPMENT

In a typical structural vibration test, only those frequency response functions (FRFs) corresponding to those co-ordinates of interest are measured. These measured FRFs can be in the form of inertance (acceleration over force $\ddot{x}/f$), mobility (velocity over force $\dot{x}/f$) or receptance (displacement over force x/f) [18]. Throughout the discussions in this paper, it is assumed that the measured FRFs are in the form of receptance data $[\boldsymbol{\alpha}(\omega)]$. What is of interest here is the development of a method which can be applied to compute structural sensitivities to given structural modifications from these measured receptances. Derivation of receptance sensitivities to given structural modifications is discussed first. Then the relationship between receptance sensitivities and those of eigenvalue and eigenvector is established from which, a method for the determination of eigenvalue and eigenvector sensitivities is then developed.

2.1. RECEPTANCE SENSITIVITY WITH RESPECT TO MASS MODIFICATION

For a practical structure, which in theory, possesses an infinite number of degrees of freedom, the receptance matrix corresponding to those co-ordinates of interest is written as,

$$[\boldsymbol{\alpha}(\omega)] = \sum_{r=1}^{\infty} \frac{\{\boldsymbol{\phi}\}_r \{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r - \omega^2}, \tag{1}$$

where λ_r and $\{\boldsymbol{\phi}\}_r$ are the rth complex eigenvalue and reduced (to those co-ordinates of interest) mass-normalised eigenvector of the structure under consideration, and ω is the excitation forcing frequency. The derivation of equation (1) is directly from the definition of the receptance corresponding to the response at co-ordinate x_i due to pure excitation applied at co-ordinate x_j, $\alpha_{ij}(\omega)$, which is given in reference [18] as,

$$\alpha_{ij}(\omega) = \sum_{r=1}^{\infty} \frac{\phi_{ir}\phi_{jr}}{\lambda_r - \omega^2}, \tag{2}$$

where ϕ_{ir} and ϕ_{jr} are the ith and jth element of the rth reduced eigenvector $\{\boldsymbol{\phi}\}_r$.

Assume that the receptance matrix $[\boldsymbol{\alpha}(\omega)]$ corresponding to those co-ordinates of interest in a specified frequency range has been measured and imagine a small mass modification Δm_i is made at a certain co-ordinate location, x_i. From the theory of mechanical coupling [18], the dynamic stiffness matrix of the original unmodified structure $[\mathbf{Z}(\omega)]$ and the dynamic stiffness matrix $[\tilde{\mathbf{Z}}(\omega)]$ of the modified structure corresponding to those measured co-ordinates of interest are related to the mass modification Δm_i by,

$$[\tilde{\mathbf{Z}}(\omega)] = [\mathbf{Z}(\omega)] - \omega^2 \Delta m_i [\mathbf{E}_{ii}], \tag{3}$$

where $[\mathbf{E}_{ii}]$ is a matrix with its (i, i) element to be unity and all the others zero in the case of mass-spring system models. A more complicated form of $[\mathbf{E}_{ii}]$ in the case of mass modification of practical structures will be discussed later.

Upon inverting both sides of equation (3) and letting $[\tilde{\mathbf{Z}}(\omega)]^{-1} = [\tilde{\boldsymbol{\alpha}}(\omega)]$ and $[\mathbf{Z}(\omega)]^{-1} = [\boldsymbol{\alpha}(\omega)]$ by definition, the following relationship in terms of receptance matrix $[\tilde{\boldsymbol{\alpha}}(\omega)]$ of the modified structure and the receptance matrix $[\boldsymbol{\alpha}(\omega)]$ of the original structure can be established,

$$[\tilde{\boldsymbol{\alpha}}(\omega)] = [[\boldsymbol{\alpha}(\omega)]^{-1} - \omega^2 \Delta m_i [\mathbf{E}_{ii}]]^{-1}. \tag{4}$$

By taking $[\boldsymbol{\alpha}(\omega)]$ out of the brackets, equation (4) can be rewritten as,

$$[\tilde{\boldsymbol{\alpha}}(\omega)] = [[\mathbf{I}] - \omega^2 \Delta m_i [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}]]^{-1}[\boldsymbol{\alpha}(\omega)]. \tag{5}$$

since Δm_i is assumed to be small, then $\|\omega^2 \Delta m_i [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}]\| \ll \|[\mathbf{I}]\|$, based on the theory of matrix perturbation [19], one has,

$$[[\mathbf{I}] - \omega^2 \Delta m_i [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}]^{-1} = [\mathbf{I}] + \sum_{k=1}^{\infty} \omega^{2k} (\Delta m_i [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}])^k, \tag{6}$$

which is a convergent matrix power series. Upon substitution of equation (6) into equation (5), one has,

$$\begin{aligned}[\tilde{\boldsymbol{\alpha}}(\omega)] &= [[\mathbf{I}] + \sum_{k=1}^{\infty} \omega^{2k} ((\Delta m_i)[\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}])^k][\boldsymbol{\alpha}(\omega)] \\ &= [\boldsymbol{\alpha}(\omega)] + \omega^2 \Delta m_i [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}][\boldsymbol{\alpha}(\omega)] + [O((\Delta m_i)^2)],\end{aligned} \tag{7}$$

where $[O((\Delta m_i)^2)]$ denotes a matrix whose elements are of the same order as $(\Delta m_i)^2$. From equation (7), receptance sensitivities with respect to mass modification Δm_i, can be calculated as,

$$\partial[\boldsymbol{\alpha}(\omega)]/\partial m_i \equiv \lim_{\Delta m_i \to 0} ([\tilde{\boldsymbol{\alpha}}(\omega)] - [\boldsymbol{\alpha}(\omega)])/\Delta m_i = \omega^2 [\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ii}][\boldsymbol{\alpha}(\omega)]. \tag{8}$$

One can see from equation (8) that once the receptance matrix $[\boldsymbol{\alpha}(\omega)]$ has been measured, receptance sensitivities can be determined.

2.2. RECEPTANCE SENSITIVITY WITH RESPECT TO STIFFNESS MODIFICATION

In the case of a stiffness modification, assuming Δk_{ij} has been made between co-ordinates x_i and x_j, then equation (3) accordingly becomes,

$$[\tilde{\mathbf{Z}}(\omega)] = [\mathbf{Z}(\omega)] + \Delta k_{ij} [\mathbf{E}_{ij}]. \tag{9}$$

Then, the receptance of the modified structure $[\tilde{\boldsymbol{\alpha}}(\omega)]$ becomes,

$$[\tilde{\boldsymbol{\alpha}}(\omega)] = [[\boldsymbol{\alpha}(\omega)]^{-1} + \Delta k_{ij} [\mathbf{E}_{ij}]^{-1}, \tag{10}$$

where $[\mathbf{E}_{ij}]$ is a matrix with its (i, i) and (j, j) elements to be 1, (i, j) and (j, i) elements to be -1 and all the others zero in the case of mass-spring system models. A more complictated form of $[\mathbf{E}_{ij}]$ for practical structures will be discussed later. Following a similar procedure as that in the case of mass modification leads to receptance sensitivities with respect to a stiffness modification as,

$$\partial[\boldsymbol{\alpha}(\omega)]/\partial k_{ij} = -[\boldsymbol{\alpha}(\omega)][\mathbf{E}_{ij}][\boldsymbol{\alpha}(\omega)]. \tag{11}$$

Again, once the receptance matrix $[\boldsymbol{\alpha}(\omega)]$ has been measured, receptance sensitivities with respect to stiffness modifications can be readily calculated based on equation (11).

2.3. EIGENVALUE AND EIGENVECTOR SENSITIVITIES

For eigenvalue sensitivities, it has been established [20] that the eigenvalue derivative of the rth mode is related to the modal parameters of the mode itself only and is expressed as equations (12a) and (12b) for mass modification Δm_i and stiffness modification Δk_{ij}, respectively,

$$\partial\lambda_r/\partial m_i = -\{\boldsymbol{\phi}\}_r^{\mathrm{T}}[\mathbf{E}_{ii}]\{\boldsymbol{\phi}\}_r, \qquad \partial\lambda_r/\partial k_{ij} = \{\phi\}_r^{\mathrm{T}}[\mathbf{E}_{ij}]\{\boldsymbol{\phi}\}_r, \tag{12a, b}$$

where $[\mathbf{E}_{ii}]$ and $[\mathbf{E}_{ij}]$ have been previously defined and the modeshape vector of the rth mode corresponding to those co-ordinates of interest $\{\boldsymbol{\phi}\}_r$ is assumed, throughout the discussion of this paper, to have been identified from modal analysis of measured receptance data. Modal analysis [18] has been very well developed in the last 30 years and provided measured receptance data are accurate, the identified modeshape vectors will, in general, be very accurate.

Having derived receptance sensitivities and eigenvalues sensitivities, and having measured and analysed receptance data to establish the eigenvalues and eigenvectors at those co-ordinates of interest, it is now possible to derive eigenvector sensitivities of the structure under study. Upon separating the contributions from those modes which are measured and from those are not measured, equation (1) can be rewritten as,

$$[\boldsymbol{\alpha}(\omega)] = \sum_{r=1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r-\omega^2} = \sum_{r=1}^{m}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r-\omega^2} + \sum_{r=m+1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r-\omega^2}, \tag{13}$$

where m is the number of identified modes. Since in the frequency range of interest, $\omega^2 \ll \mathrm{Re}(\lambda_{m+1})$, the second term on the right side of equation (13) can be written as,

$$\sum_{r=m+1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r-\omega^2} = \sum_{r=m+1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r}\left(1-\frac{\omega^2}{\lambda_r}\right)^{-1} = \sum_{r=m+1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r}\sum_{k=0}^{\infty}\left(\frac{\omega^2}{\lambda_{\mathrm{r}}}\right)^{\mathrm{k}}. \tag{14}$$

By retaining just the first order term and substituting equation (14) into equation (13), equation (13) becomes,

$$[\boldsymbol{\alpha}(\omega)] = \sum_{r=1}^{m}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r-\omega^2} + \sum_{r=m+1}^{\infty}\frac{\{\boldsymbol{\phi}\}_r\{\boldsymbol{\phi}\}_r^{\mathrm{T}}}{\lambda_r} = [\boldsymbol{\phi}[\diagdown\lambda-\omega^2\diagdown]^{-1}[\boldsymbol{\phi}]^{\mathrm{T}} + [\mathbf{R}], \tag{15}$$

where $[\diagdown\lambda\diagdown]$ and $[\boldsymbol{\phi}]$ are the m identified eigenvalues and modeshape matrix from modal analysis of the measured receptance data, $[\mathbf{R}]$ is a constant residue matrix representing the contribution of higher unmeasured modes. Though it is accurate only to a first order, such an approximation as equation (15) is indeed very accurate (as will be shown later in the numerical case studies) especially when data points around resonances are considered. To eliminate the residue term $[\mathbf{R}]$, receptance data at two different measurement frequencies $\omega = \omega_q$ and $\omega = \omega_s$ (preferably neighbouring frequency points) can be used,

$$[\boldsymbol{\alpha}(\omega_q)] - [\boldsymbol{\alpha}(\omega_s)] = [\boldsymbol{\phi}][[\diagdown(\lambda-\omega_q^2)\diagdown]^{-1} - [\diagdown(\lambda-\omega_s^2)\diagdown]^{-1}][\boldsymbol{\phi}]^{\mathrm{T}}. \tag{16}$$

Upon differentiation with respect to structural modification p_k (p_k can be mass modification Δm_i or stiffness modification Δk_{ij}), equation (16) becomes,

$$\frac{\partial[\boldsymbol{\alpha}(\omega_q)]}{\partial p_k} - \frac{\partial[\boldsymbol{\alpha}(\omega_s)]}{\partial p_k} = \frac{\partial[\boldsymbol{\phi}]}{\partial p_k}[[\diagdown(\lambda-\omega_q^2)\diagdown]^{-1} - [\diagdown(\lambda-\omega_s^2)\diagdown]^{-1}][\boldsymbol{\phi}^{\mathrm{T}}$$

$$+[\boldsymbol{\phi}][[\diagdown(\lambda-\omega_q^2)\diagdown]^{-1} - [\diagdown(\lambda-\omega_s^2)\diagdown]^{-1}]\frac{\partial[\boldsymbol{\phi}]^{\mathrm{T}}}{\partial p_k}$$

$$+[\boldsymbol{\phi}][[\diagdown(\lambda-\omega_s^2)\diagdown]^{-2} - [\diagdown(\lambda-\omega_q^2)\diagdown]^{-2}]\frac{\partial[\diagdown\lambda\diagdown]}{\partial p_k}[\boldsymbol{\phi}]^{\mathrm{T}}. \tag{17}$$

Substituting the known eigenvalue sensitivites of equations (12a) or (12b) into equation (17) and rearranging, gives.

$$[\mathbf{Z}][A_{qs}]+[A_{qs}]^{\mathrm{T}}[\mathbf{Z}]^{\mathrm{T}} = [B_{qs}], \tag{18}$$

where $[A_{qs}]$, $[B_{qs}]$ and $[\mathbf{Z}]$ are defined as,

$$[B_{qs}]=\frac{\partial[\boldsymbol{\alpha}(\omega_q)]}{\partial p_k} - \frac{\partial[\boldsymbol{\alpha}(\omega_s)]}{\partial p_k} - [\boldsymbol{\phi}][[\diagdown(\lambda-\omega_s^2)\diagdown]^{-2} - [\diagdown(\lambda-\omega_q^2)\diagdown]^{-2}]\frac{\partial[\diagdown\lambda\diagdown]}{\partial p_k}[\boldsymbol{\phi}]^{\mathrm{T}},$$

$$[A_{qs}]=[[\diagdown(\lambda-\omega_q^2)\diagdown]^{-1} - [\diagdown(\lambda-\omega_s^2)\diagdown]^{-1}][\boldsymbol{\phi}]^{\mathrm{T}}, \quad [\mathbf{Z}] = \partial[\boldsymbol{\phi}]/\partial p_k\ . \tag{19}$$

Equation (17) represents the relationship between receptance sensitivities and eigenvector sensitivities. Once receptance sensitivities are known, eigenvector sensitivities can be computed from equation (17). Also, it has to be mentioned that the number of co-ordinates measured has, in theory, no effect upon the accuracy of computed eigenvector derivatives, provided those co-ordinates associated with the points where structural modifications are of interest have been measured. From equation (18), when two sets of receptance data at two frequency points are used, eigenvector derivatives $[\mathbf{Z}]$ can be solved. However, the solution for $[\mathbf{Z}]$ from equation (18) can sometimes be numerically difficult to achieve due to the non-symmetric nature of $[\mathbf{Z}][A_{qs}]$, especially when the number of measured co-ordinates and number of modes of interest become large. To overcome this difficulty, it is recommended that the eigenvector sensitivities of each individual mode be solved one at a time in the case of separated modes, and a pair of modes at a time in the case of two close modes. This is discussed next.

2.4. SEPARATE MODE

For measurement frequency ω around the rth resonance $\omega \approx \omega_r$, the ith column of the measured receptance matrix $[\boldsymbol{\alpha}(\omega)]$ can be written as,

$$\{\boldsymbol{\alpha}(\omega)\}_i = (\boldsymbol{\phi}_{ir}/\lambda_r - \omega^2)\{\boldsymbol{\phi}\}_r + \{\mathbf{R}_r\}, \tag{20}$$

where the contribution of all the other modes to the mode under consideration is considered as a constant residual vector $\{\mathbf{R}_r\}$. Considering two frequency points $\omega = \omega_q$ and $\omega = \omega_s$ around the resonance and eliminating the residue term $\{\mathbf{R}_r\}$, one has,

$$\{\boldsymbol{\alpha}(\omega_q)\}_i - \{\boldsymbol{\alpha}(\omega_s)\}_i = (1/(\lambda_r-\omega_q^2) - 1/(\lambda_r-\omega_s^2))\phi_{ir}\{\boldsymbol{\phi}\}_r \tag{21}$$

Upon differentiation of equation (21) with respect to p_k, one has,

$$\frac{\partial\{\boldsymbol{\alpha}(\omega_q)\}_i}{\partial p_k} - \frac{\partial\{\boldsymbol{\alpha}(\omega_s)\}_i}{\partial p_k} = \left(\frac{1}{[\lambda_r-\omega_s^2]^2} - \frac{1}{[\lambda_r-\omega_q^2]^2}\right)\frac{\partial\lambda_r}{\partial p_k}\phi_{ir}\{\boldsymbol{\phi}\}_r$$

$$+\left(\frac{1}{\lambda_r-\omega_q^2}-\frac{1}{\lambda_r-\omega_s^2}\right)\frac{\partial\phi_{ir}}{\partial p_k}\{\boldsymbol{\phi}\}_r$$

$$+\left(\frac{1}{\lambda_r-\omega_q^2}-\frac{1}{\lambda_r-\omega_s^2}\right)\phi_{ir}\frac{\partial\{\boldsymbol{\phi}\}_r}{\partial p_k} \tag{22}$$

By substituting known eigenvalue derivative $\partial\lambda_r/\partial p_k$, equation (22) can be transformed as,

$$\beta_{qs}\{\mathbf{Z}\}+\{\mathbf{U}_{qs}\}z_i=\{\mathbf{V}_{qs}\}, \tag{23}$$

where $\{\mathbf{V}_{qs}\}$, $\{\mathbf{U}_{qs}\}$, β_{qs} and $\{\mathbf{Z}\}$ are defined as,

$$\{\mathbf{V}_{qs}\}=\frac{\partial\{\boldsymbol{\alpha}(\omega_q)\}_i}{\partial p_k}-\frac{\partial\{\boldsymbol{\alpha}(\omega_s)\}_i}{\partial p_k}\Bigg)-\left(\frac{1}{[\lambda_r-\omega_s^2]^2}-\frac{1}{[\lambda_r-\omega_q^2]^2}\right)\frac{\partial\lambda_r}{\partial p_k}\phi_{ir}\{\boldsymbol{\phi}\}_r$$

$$\{\mathbf{U}_{qs}\}=\left(\frac{1}{\lambda_r-\omega_q^2}-\frac{1}{\lambda_r-\omega_s^2}\right)\frac{\partial\phi_{ir}}{\partial p_k}\{\boldsymbol{\phi}\}_r,\qquad \beta_{qs}=\left(\frac{1}{\lambda_r-\omega_q^2}-\frac{1}{\lambda_r-\omega_s^2}\right)\phi_{ir},$$

$$\{\mathbf{Z}\}=\partial\{\boldsymbol{\phi}\}_r/\partial p_k \tag{24}$$

The vectors $\{\mathbf{V}_{qs}\}$, $\{\mathbf{U}_{qs}\}$ and the scalar β_{qs} are all known since the eigenvalues and eigenvectors involved have ben identified from the process of modal analysis of measured receptance data. The procedure for the solution of $\{\mathbf{Z}\}$ from equation (23) is straightforward by first solving z_i as,

$$\beta_{qs}z_i+U_{qs}^{(i)}z_i=V_{qs}^{(i)}\Rightarrow z_i=V_{qs}^{(i)}/(U_{qs}^{(i)}+\beta_{qs}), \tag{25}$$

where $U_{qs}^{(i)}$ is defined as the ith element of $\{\mathbf{U}_{qs}\}$. Once z_i is solved, $\{\mathbf{Z}\}$ can be easily computed from equation (23) by substituting known values of z_i. When more measured receptance data at different pairs of frequencies around the resonance are used, least squares procedure [21] can be employed to solve for $\{\mathbf{Z}\}$. Assume L different sets of receptance data around the rth mode are used, the least squares solution of equation (23) becomes,

$$\sum_{m'=1}^{L}(\beta_{qs}^{(m')2})\{\mathbf{Z}\}+\sum_{m'=1}^{L}\beta_{qs}^{(m')}\{\mathbf{U}_{qs}\}^{(m')}z_i=\sum_{m'=1}^{L}\beta_{qs}^{(m')}\{\mathbf{V}_{qs}\}^{(m')} \tag{26}$$

where index m' denotes m'th data set. From equation (26), $\{\mathbf{Z}\}$ can be solved in the same way as discussed. A least-squares procedure is important to improve the solution accuracy in the practical case where measured data are contaminated by random measurement errors.

2.5. CLOSE MODES

In the case of close modes, receptance data around the two close resonances $\omega\approx\omega_r$ can be expressed as,

$$\{\boldsymbol{\alpha}(\omega)\}_i=\frac{\phi_{ir}}{\lambda_r-\omega^2}\{\boldsymbol{\phi}\}_r+\frac{\phi_{ir'}}{\lambda_{r'}-\omega^2}\{\boldsymbol{\phi}\}_{r'}+\{\mathbf{R}_r\},\qquad (r'=r+1), \tag{27}$$

where the contribution of all the other modes to the two modes under consideration is considered as a constant residual vector $\{\mathbf{R}_r\}$. Similarly, considering receptant data at two different measurement frequencies around the resonances to eliminate the residue term, one has,

$$\{\boldsymbol{\alpha}(\omega_q)\}_i - \{\boldsymbol{\alpha}(\omega_s)\}_i = \left(\frac{1}{\lambda_r - \omega_q^2} - \frac{1}{\lambda_r - \omega_s^2}\right)\phi_{ir}\{\boldsymbol{\phi}\}_r + \left(\frac{1}{\lambda_{r'} - \omega_q^2} - \frac{1}{\lambda_{r'} - \omega_s^2}\right)\phi_{ir'}\{\boldsymbol{\phi}\}_{r'}. \tag{28}$$

Upon differentiation with respect to p_k, equation (28) becomes,

$$\begin{aligned}\frac{\partial\{\boldsymbol{\alpha}(\omega_q)\}_i}{\partial p_k} - \frac{\partial\{\boldsymbol{\alpha}(\omega_s)\}_i}{\partial p_k} = &\left(\frac{1}{[\lambda_r - \omega_s^2]^2} - \frac{1}{[\lambda_r - \omega_q^2]^2}\right)\frac{\partial\lambda_r}{\partial p_k}\,\phi_{ir}\{\boldsymbol{\phi}\}_r \\ &+ \left(\frac{1}{[\lambda_{r'} - \omega_s^2]^2} - \left(\frac{1}{[\lambda_{r'} - \omega_q^2]^2}\right)\frac{\partial\lambda_{r'}}{\partial p_k}\,\phi_{ir'}\{\boldsymbol{\phi}\}_{r'} \\ &+ \left(\frac{1}{\lambda_r - \omega_q^2} - \frac{1}{\lambda_r - \omega_s^2}\right)\frac{\partial\phi_{ir}}{\partial p_k}\{\boldsymbol{\phi}\}_r \\ &+ \left(\frac{1}{\lambda_{r'} - \omega_q^2} - \left(\frac{1}{\lambda_{r'} - \omega_s^2}\right)\frac{\partial\phi_{ir'}}{\partial p_k}\{\boldsymbol{\phi}\}_{r'} \\ &+ \left(\frac{1}{\lambda_r - \omega_q^2} - \frac{1}{\lambda_r - \omega_s^2}\right)\phi_{ir}\frac{\partial\{\boldsymbol{\phi}\}_r}{\partial p_k} \\ &+ \left(\frac{1}{\lambda_{r'} - \omega_q^2} - \frac{1}{\lambda_{r'} - \omega_s^2}\right)\phi_{ir'}\frac{\partial\{\boldsymbol{\phi}\}_{r'}}{\partial p_k}\end{aligned} \tag{29}$$

After some mathematical manipulation, equation (29) can be rewritten as,

$$\begin{aligned}\{\mathbf{V}\} &= \{\mathbf{U}_1\}z_{1i} + \beta_1\{\mathbf{Z}_1\} + \{\mathbf{U}_2\}z_{2i} + \beta_2\{\mathbf{Z}_2\} \\ &= \beta_1[\mathbf{A}_1]\{\mathbf{Z}_1\} + \beta_2[\mathbf{A}_2]\{\mathbf{Z}_2\} = [\beta_1[\mathbf{A}_1]\;\beta_2[\mathbf{A}_2]]\begin{Bmatrix}\{\mathbf{Z}_1\}\\ \{\mathbf{Z}_2\}\end{Bmatrix} = [\mathbf{A}]\{\mathbf{Z}\},\end{aligned} \tag{30}$$

where β_1, β_2, $\{\mathbf{V}\}$, $\{\mathbf{Z}_1\}$, $\{\mathbf{Z}_2\}$, $\{\mathbf{U}_1\}$, $\{\mathbf{U}_2\}$, $[\mathbf{A}_1]$ and $[\mathbf{A}_2]$ are defined as,

$$\beta_1 = \left(\frac{1}{\lambda_r - \omega_q^2} - \frac{1}{\lambda_r - \omega_s^2}\right)\phi_{ir}, \qquad \beta_2 = \left(\frac{1}{\lambda_{r'} - \omega_q^2} - \frac{1}{\lambda_{r'} - \omega_s^2}\right)\phi_{ir'},$$

$$\{\mathbf{V}\} = \frac{\partial\{\boldsymbol{\alpha}(\omega_q)\}_i}{\partial p_k} - \frac{\partial\{\boldsymbol{\alpha}(\omega_s)\}_i}{\partial p_k} - \left(\frac{1}{[\lambda_r - \omega_s^2]^2} - \frac{1}{[\lambda_r - \omega_q^2]}^2\right)\frac{\partial\lambda_r}{\partial p_k}\,\phi_{ir}\{\boldsymbol{\phi}\}_r$$

$$-\left(\frac{1}{[\lambda_{r'} - \omega_s^2]^2} - \frac{1}{[\lambda_{r'} - \omega_q^2]^2}\right)\frac{\partial\lambda_{r'}}{\partial p_k}\,\phi_{ir'}\{\boldsymbol{\phi}\}_{r'}, \qquad \{\mathbf{Z}_1\} = \frac{\partial\{\boldsymbol{\phi}\}_r}{\partial p_r}, \qquad \{\mathbf{Z}_2\} = \frac{\partial\{\boldsymbol{\phi}\}_{r'}}{\partial p_{r'}},$$

$$\{\mathbf{U}_1\} = \left(\frac{1}{\lambda_r - \omega_q^2} - \frac{1}{\lambda_r - \omega_s^2}\right)\{\boldsymbol{\phi}\}_r, \qquad \{\mathbf{U}_2\} = \left(\frac{1}{\lambda_{r'} - \omega_q^2} - \frac{1}{\lambda_{r'} - \omega_s^2}\right)\{\boldsymbol{\phi}\}_{r'},$$

$$[\mathbf{A}_1] = [\mathbf{I}] + [\{\mathbf{0}\}_1, \ldots, \{\mathbf{0}\}_{i-1}\{\mathbf{U}_1\}/\beta_1\{\mathbf{0}\}_{i+1}, \ldots, \{\mathbf{0}\}_n],$$

$$[\mathbf{A}_2] = [\mathbf{I}] + [\{\mathbf{0}\}_1, \ldots, \{\mathbf{0}\}_{i-1}\{\mathbf{U}_2\}/\beta_2\{\mathbf{0}\}_{i+1}, \ldots, \{\mathbf{0}\}_n]. \tag{31}$$

In principle, only two sets of receptance data are needed in order to solve for $\{\mathbf{Z}\}$ and hence $\{\mathbf{Z}_1\}$ and $\{\mathbf{Z}_2\}$ from equation (30). When L sets of receptance data around the resonances are used, equation (30) can be solved using least-squares as,

$$\left[\sum_{m'=1}^{L}[\mathbf{A}^{(m')}]^{\mathrm{T}}[\mathbf{A}^{(m')}]\right]\{\mathbf{Z}\} = \left\{\sum_{m'=1}^{L}[\mathbf{A}^{(m')}]^{\mathrm{T}}\{\mathbf{V}^{(m')}\}\right\}. \tag{32}$$

Again, such least-squares procedure becomes important in improving solution accuracy in cases where measured receptance and identified modal data are contaminated by random measurement errors.

3. NUMERICAL ASSESSMENT

A method has so far been developed to compute sensitivities of receptance and eigenvalues and eigenvectors from vibration test data. In order to assess the practical applicability of the proposed method, extensive numerical simulations have been conducted to test its numerical accuracy, the validity of the assumption made in deriving eigenvector sensitivities and the effect of random measurement errors. The system model used is a lumped parameter model shown in Figure 1 which represents a practical bladed disc system [22]. The number of blades is assumed to be 24 and the values for the parameters used in the study are shown in Figure 1. The equations of motion for the ith blade and ith disc sector are,

$$m\ddot{x}_{2i-1} + c(\dot{x}_{2i-1} - \dot{x}_{2i}) + k(x_{2i-1} - x_{2i}) = 0,$$

$$M\ddot{x}_{2i} + k(x_{2i} - x_{2i-1}) + k_g x_{2i} + K(2x_{2i} - x_{2i-2} - x_{2i+2}) = 0. \tag{33}$$

From equation (33), system mass, stiffness and damping matrices can be assembled by considering all the blade and disc sectors ($i = 1, 24$). The values of k_g are chosen according to equation (34) that the first mode of the system becomes a non-rigid body mode and

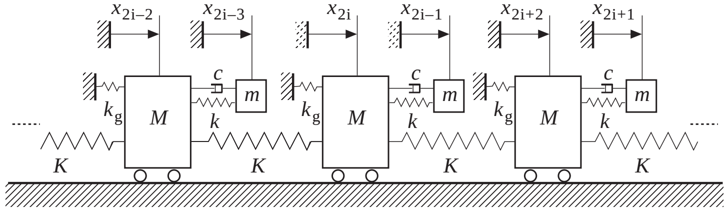

Figure 1. A 24 bladed disc lumped parameter model, $M = 0{\cdot}7$ kg, $m = 0{\cdot}3$ kg, $K = 2{\cdot}4\mathrm{E}7$ N/M, $k = 2{\cdot}6\mathrm{E}6$ N/M.

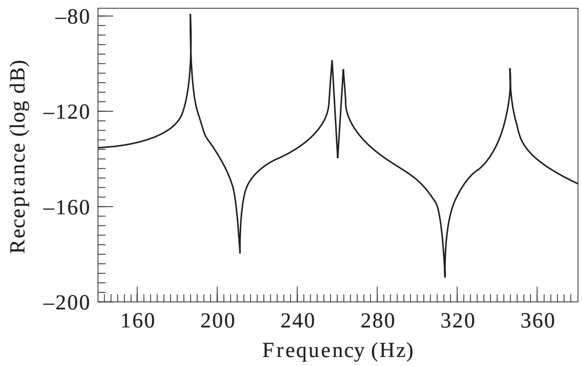

Figure 2. Typical point receptance of the system $\alpha_{22}(\omega)$.

the second and the third become close modes instead of repeated ones as shown in the point receptance plot of co-ordinate x_2 of Figure 2,

$$\begin{cases} k_g = 1{\cdot}0\ \text{E7 N/m} & (i = 1, 7, 13), \\ k_g = 1{\cdot}3\ \text{E7 N/m} & (i = 19), \\ k_g = 0\ \text{N/m} & \text{otherwise.} \end{cases} \tag{34}$$

To simulate the practical case where generally only receptances at a few co-ordinates of interest are measured at a certain frequency range, 12 co-ordinates x_2, x_4, x_5, x_8, x_{11}, x_{16}, x_{19}, x_{24}, x_{29}, x_{34}, x_{39}, x_{44} (out of 48 in total) are assumed to have been measured in this numerical case study. Three different cases are devised to fully test the practicality of the proposed method: (i) undamped separate and close modes, (ii) damped modes and (iii) receptance data with measurement errors. Before embarking on these case studies, it is necessary to verify first the assumption made in equations (20) and (27) that the contribution of other modes to the receptance of the mode or modes under consideration $\{R_r\}|_{\omega \approx \omega_r}$ is constant. For receptance data around the first mode, the contribution of the first mode itself and that of other modes to the point receptance of co-ordinate x_2 are calculated and shown in Figure 3. From Figure 3, one can see that the contribution of other modes is really very much a constant and is very small (the difference is more than 35 dB which is less than 1·8%). Also, one might have noticed that in the derivation of eigenvector sensitivities, it is implicitly assumed that the derivative of the residue with

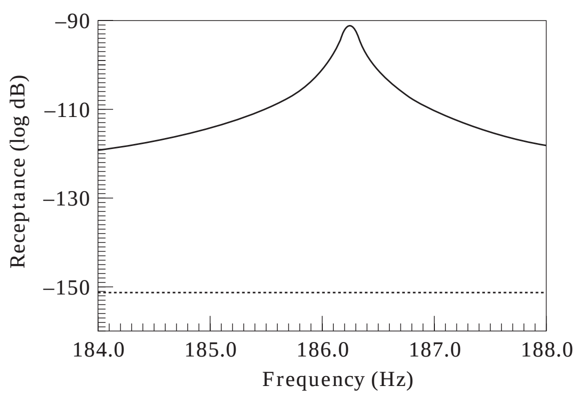

Figure 3. Comparison of contributions of the mode and the residue: ——, 20 log $|\alpha_{22}(\omega) - R_1(\omega)|$; ....., 20 log $|R_1(\omega)|$.

respect to design parameter $\partial\{R_r\}/\partial p_k|_{\omega\approx\omega_r}$ should also be a constant since from equation (20), one has the following relationship by taking differentiation with respect to p_k,

$$\frac{\partial\{\boldsymbol{\alpha}(\omega)\}_i}{\partial p_k} = -\frac{1}{[\lambda_r-\omega^2]^2}\frac{\partial\lambda_r}{\partial p_k}\phi_{ir}\{\boldsymbol{\phi}\}_r + \frac{1}{\lambda_r-\omega^2}\frac{\partial\phi_{ir}}{\partial p_k}\{\boldsymbol{\phi}\}_r + \frac{1}{\lambda_r-\omega^2}\phi_{ir}\frac{\partial\{\boldsymbol{\phi}\}_r}{\partial p_k} + \frac{\partial\{R_r(\omega)\}}{\partial p_k}. \tag{35}$$

Considering the receptance sensitivities $\omega=\omega_q$ and $\omega=\omega_s$ and taking their difference, one has,

$$\frac{\partial\{\boldsymbol{\alpha}(\omega_q)\}_i}{\partial p_k} - \frac{\partial\{\boldsymbol{\alpha}(\omega_s)\}_i}{\partial p_k} = \left(\frac{1}{[\lambda_r-\omega_s^2]^2} - \frac{1}{[\lambda_r-\omega_q^2]^2}\right)\frac{\partial\lambda_r}{\partial p_k}\phi_{ir}\{\boldsymbol{\phi}\}_r + \left(\frac{1}{\lambda_r-\omega_q^2} - \frac{1}{\lambda_r-\omega_s^2}\right)\frac{\partial\phi_{ir}}{\partial p_k}\{\boldsymbol{\phi}\}_r + \left(\frac{1}{\lambda_r-\omega_q^2} - \frac{1}{\lambda_r-\omega_s^2}\right)\phi_{ir}\frac{\partial\{\boldsymbol{\phi}\}_r}{\partial p_k} + \frac{\partial\{\mathbf{R}_r(\omega_q)\}}{\partial p_k} - \frac{\partial\{\mathbf{R}_r(\omega_s)\}}{\partial p_k} \tag{36}$$

By comparing equation (36) with equation (22), it is not difficult to see that an assumption of constant $\partial\{\mathbf{R}_r\}/\partial p_k$ was made during the derivation of equation (22). Indeed, when data around resonances are considered, $\partial\{\mathbf{R}_r\}/\partial p_k$ remains very much a constant vector and its magnitudes are very small when compared with those of the receptance sensitivities as is illustrated in Figure 4.

The case of an undamped system is investigated first by setting the damping coefficient c in Figure 1 to be 0. Receptance sensitivities with respect to the mass change of co-ordinate x_8 are calculated based on equation (8) by using the "measured" receptance terms and the receptance sensitivity $\partial\alpha_{22}(\omega)/\partial m_8$ is given in Figure 5 which shows similar characteristics to those of receptance but with larger dynamic range, since receptance sensitivity is given by the product of receptances as shown in equation (8). By applying the proposed method and using receptance sensitivity data points around the resonance under study, eigenvector sensitivities corresponding to the "measured" co-ordinates of interest with respect to the

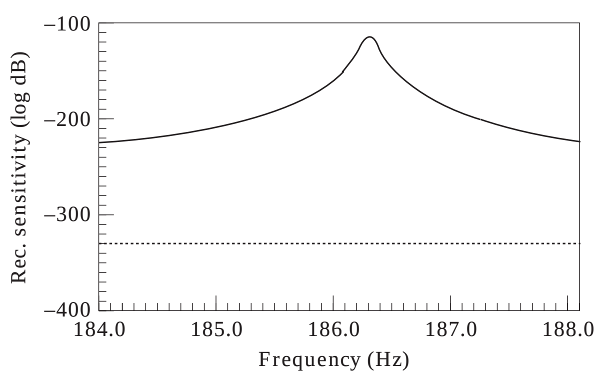

Figure 4. Comparison of derivatives of the mode and the residue: ——, 20 log $|(\partial\alpha_{22}(\omega)/\partial p_k)-(\partial R_1(\omega)/\partial p_k)|$; ⋯⋯, 20 log $|(\partial R_1(\omega)/\partial p_k)|$.

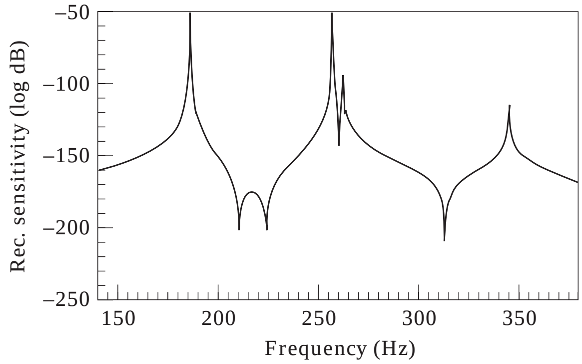

Figure 5. Receptance sensitivity of undamped system $\partial\alpha_{22}(\omega)/\partial m_8$.

mass change Δm_8 of co-ordinate x_8 and the stiffness change Δk_{24} between co-ordinates x_2 and x_4 are calculated for the first (separate) and second (close) modes. The results are shown in Table 1 and Table 2 together with comparisons with their exact values which are computed based on the formula proposed by Fox and Kapoor [12]. It can be seen from these results that the proposed method is indeed very accurate.

Practical structures possess some degree of damping and in order to assess the effect of damping upon the numerical accuracy of the proposed method, viscous damping is added to the system by setting $c = 25{\cdot}0$ Ns/m which is supposed to represent the energy dissipation mechanism due to friction at the blade disc root connections. In this case, the receptance of the system becomes complex and so is the receptance sensitivity. Again, receptance sensitivities are calculated and $\partial\alpha_{22}(\omega)/\partial m_8$ is given in Figure 6 which shows that for some modes, the real part of the sensitivity drops significantly around the resonance due to the domination of the imaginary part of receptance when damping exists. The proposed method is applied to calculate the eigenvector derivatives of mode 2 and the results are shown in Table 3. Again, the results are very accurate for all the real parts which are always dominant. For some imaginary parts, a few percent error has been introduced due to their small magnitudes, but the results are generally quite accurate.

TABLE 1

Predicted and exact eigenvector sensitivities of mode 1 (*undamped case*)

Co-ord. number	Predicted $\partial\{\phi\}_1/\partial m_8$	Exact $\partial\{\phi\}_1/\partial m_8$	error (%)	Predicted $\partial\{\phi\}_1/\partial k_{24}$	Exact $\partial\{\phi\}_1/\partial k_{24}$	error (%)
x_2	0·455134 E-2	0·455283 E-2	**0·03274**	0·327367 E-9	0·327398 E-9	0·00947
x_4	0·115614 E-1	0·115617 E-1	0·00259	−0·103066 E-8	−0·103067 E-8	0·00097
x_5	0·195798 E-1	0·195825 E-1	0·01378	−0·110410 E-8	−0·110408 E-8	0·00181
x_8	0·249898 E-1	0·249903 E-1	0·00200	−0·845672 E-9	−0·845665 E-9	0·00083
x_{11}	0·103179 E-1	0·103176 E-1	0·00291	−0·563145 E-9	−0·563120 E-9	0·00444
x_{16}	−0·260231 E-2	−0·260291 E-2	0·02305	−0·206142 E-9	−0·206173 E-9	**0·01504**
x_{19}	−0·159615 E-1	−0·159649 E-1	0·02130	0·122488 E-10	0·122492 E-10	0·00327
x_{24}	−0·146539 E-1	−0·146559 E-1	0·01365	0·121591 E-9	0·121581 E-9	0·00822
x_{29}	−0·261210 E-1	−0·261235 E-1	0·00957	0·443812 E-9	0·443780 E-9	0·00721
x_{34}	−0·190231 E-1	−0·190241 E-1	0·00526	0·398103 E-9	0·398095 E-9	0·00201
x_{39}	−0·151721 E-1	−0·151733 E-1	0·00791	0·520345 E-9	0·520301 E-9	0·00846
x_{44}	−0·794764 E-2	−0·794810 E-2	0·00579	0·494681 E-9	0·494633 E-9	0·00970

TABLE 2

Predicted and exact eigenvector sensitivities of mode 2 (undamped case)

Co-ord. number	Predicted $\partial\{\phi\}_2/\partial m_8$	Exact $\partial\{\phi\}_2/\partial m_8$	error (%)	Predicted $\partial\{\phi\}_2/\partial k_{24}$	Exact $\partial\{\phi\}_2/\partial k_{24}$	error (%)
x_2	−0·0321394	−0·0321460	0·02053	−0·808732 E-9	0·809075 E-9	0·04239
x_4	0·0406503	0·0406481	0·00546	−0·188998 E-8	−0·189025 E-8	0·01428
x_5	0·1514802	0·1514478	0·02139	−0·448480 E-8	−0·448257 E-8	0·04975
x_8	0·1666576	0·1666498	0·00468	−0·405940 E-8	−0·405845 E-8	0·02341
x_{11}	0·2509612	0·2509326	0·01140	−0·617908 E-8	−0·617810 E-8	0·01586
x_{16}	0·1666809	0·1666777	0·00192	−0·407641 E-8	−0·407603 E-8	0·00932
x_{19}	0·2023554	0·2023333	0·01092	−0·489879 E-8	−0·489903 E-8	0·00490
x_{24}	0·0391543	0·0391724	0·04621	−0·106506 E-8	−0·106340 E-8	**0·15610**
x_{29}	−0·2122185	−0·2123515	0·06263	0·471875 E-8	0·472019 E-8	0·03051
x_{34}	−0·2140478	−0·2141323	0·03946	0·471314 E-8	0·471696 E-8	0·08098
x_{39}	−0·3172522	−0·3172797	0·01440	0·690309 E-8	0·690379 E-8	0·01014
x_{44}	−0·2217501	−0·2217606	0·00473	0·484400 E-8	0·484381 E-8	0·00392

Since the proposed method is intended for the application of experimental determination of structural sensitivities from test data which are inevitably contaminated by measurement errors, it becomes necessary to evaluate the performance of the method when measurement errors are involved. In this study, 3% random noise is added to the "measured" receptances and the derived eigenvectors and 0·5% random errors are added to the derived eigenvalues considering the fact that natural frequencies can be more accurately identified. In this case, it is recommended that more data points should be used in order to minimize the effect of measurement errors using least squares to improve solution accuracy. For mode 2, 20 frequency data points ($L = 10$) around the resonance are used to estimate the eigenvector sensitivities based on equation (32) and the results are shown in Table 4. It is found that when more than 10 sets of frequency data ($L \geqslant 10$) are used, the improvement on the accuracy of the results obtained is negligible. From Table 4, it can be seen that the results obtained are quite accurate and should perhaps be considered adequate for practical applications.

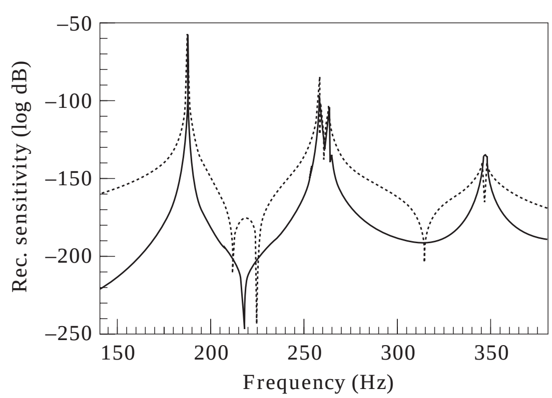

Figure 6. Receptance sensitivity of damped system $\partial\alpha_{22}(\omega)/\partial m_8$: ·····, Re $[\partial\alpha_{22}(\omega)/\partial m_8]$; ——, Im $[\partial\alpha_{22}(\omega)/m_8]$.

TABLE 3

Predicted and exact eigenvector sensitivities of mode 2 (damped case)

Co-ord. number	Predicted $\mathrm{Re}[\partial\phi_2/\partial m_8]$	Exact $\mathrm{Re}[\partial\phi_2/\partial m_8]$	error (%)	Predicted $\mathrm{Im}[\partial\phi_2/\partial m_8]$	Exact $\mathrm{Im}[\partial\phi_2/\partial m_8]$	error (%)
x_2	−0·0321485	−0·0321541	0·01742	−0·330897 E-3	−0·328685 E-3	0·67298
x_4	0·0406631	0·0406542	0·02189	0·226937 E-3	0·231104 E-3	1·80308
x_5	0·1514723	0·1514466	0·01694	−0·210193 E-3	−0·196594 E-3	6·91730
x_8	0·1666873	0·1666812	0·00366	0·122602 E-2	0·122944 E-2	0·02782
x_{11}	0·2509455	0·2509311	0·00574	−0·359217 E-3	−0·348249 E-3	3·13614
x_{16}	0·1667298	0·1667116	0·01038	0·133161 E-2	0·133834 E-2	0·50286
x_{19}	0·2023528	0·2023319	0·01033	−0·306028 E-3	−0·300852 E-3	1·72044
x_{24}	0·0391935	0·0391823	0·02858	0·378519 E-3	0·395044 E-3	4·36569
x_{29}	−0·2122403	−0·2123499	**0·05161**	0·253883 E-3	0·282579 E-3	**10·1550**
x_{34}	−0·2141034	−0·2141737	0·03282	−0·165245 E-2	−0·162185 E-2	1·88673
x_{39}	−0·3172545	−0·3172782	0·00237	0·445333 E-3	0·453962 E-3	1·90081
x_{44}	−0·2217971	−0·2218066	0·00428	−0·182010 E-2	−0·181743 E-2	0·14670

4. EXPERIMENTAL INVESTIGATION

The method has been rigorously assessed based on extensive numerical case studies and here in this section, an experimental investigation is carried out and some considerations regarding the practical applications of the proposed method are highlighted. The test structure considered is an aluminum beam with rectangular cross-section as shown in Figure 7. The measurement set-up is schematically shown in Figure 8. Sine sweep testing was used to measure the required receptance data since it is the most accurate measurement technique to date due to its high signal to noise ratio [18]. The whole measurement process is controlled by an in-house software in which measurement parameters such as starting frequency, finishing frequency, number of frequency points, integration time, delay time, measurement accuracy, vibration and excitation amplitudes can be set appropriately. The excitation point was chosen to be co-ordinate x_3 and 6 frequency response functions (receptances) were measured along the beam and are shown in Figure 9. In the measured frequency range, 3 bending modes are clearly identified. Based on equation (8), one typical sensitivity of receptance $\alpha_{33}(\omega)$ with respect to mass modification at co-ordinate x_3 is

TABLE 4

Predicted and exact eigenvector sensitivities of mode 2 (polluted data)

Co-ord. number	Predicted $\partial\{\boldsymbol{\phi}\}_2/\partial m_8$	Exact $\partial\{\boldsymbol{\phi}\}_2/\partial m_8$	error (%)	Predicted $\partial\{\boldsymbol{\phi}\}_2/\partial k_{24}$	Exact $\partial\{\boldsymbol{\phi}\}_2/\partial k_{24}$	error (%)
x_2	0·501495 E-2	0·455283 E-2	10·1501	0·298740 E-9	0·327398 E-9	8·75312
x_4	0·109609 E-1	0·115617 E-1	5·19638	−0·111348 E-8	−0·103067 E-8	8·03412
x_5	0·191711 E-1	0·195825 E-1	2·10097	−0·129286 E-8	−0·110408 E-8	**17·0986**
x_8	0·284827 E-1	0·249903 E-1	13·9752	−0·823295 E-9	−0·845665 E-9	2·64523
x_{11}	0·101560 E-1	0·103176 E-1	1·56732	−0·581438 E-9	−0·563120 E-9	3·25289
x_{16}	−0·218911 E-2	−0·260291 E-2	**15·8976**	−0·224654 E-9	−0·206173 E-9	8·96372
x_{19}	−0·167404 E-1	−0·159649 E-1	4·85765	0·106887 E-10	0·122492 E-10	12·7399
x_{24}	−0·130293 E-1	−0·146559 E-1	11·0984	0·131278 E-9	0·121581 E-9	7·97621
x_{29}	−0·284885 E-1	−0·261235 E-1	9·05312	0·513756 E-9	0·443780 E-9	15·7682
x_{34}	−0·176102 E-1	−0·190241 E-1	7·43211	0·389356 E-9	0·398095 E-9	2·18634
x_{39}	−0·145686 E-1	−0·151733 E-1	3·98518	0·548084 E-9	0·520301 E-9	5·33986
x_{44}	−0·904240 E-2	−0·794810 E-2	13·7681	0·474920 E-9	0·494633 E-9	3·98534

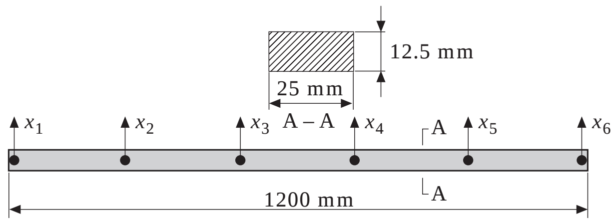

Figure 7. An aluminum beam structure.

computed as shown in Figure 10 which exhibits similar characteristics as those of the lumped parameter model. The measured receptances were then analysed using ICATS commercial modal analysis software [23] to establish the eigenvalues and eigenvectors of the structure. These eigenvalues and eigenvectors were then employed together with the receptance sensitivities to compute the eigenvector sensitivities required based on the proposed method. Eigenvector sensitivities of all the 3 modes with respect to mass modification at co-ordinate x_3 were derived and they are shown in Table 5. The imaginary parts of the eigenvector sensitivities are very small as compared with their real parts and this makes sense since in theory, eigenvector sensitivities of a uniform beam should be virtually real since the damping distribution of the beam is virtually proportional and hence its eigenvectors are real.

The method has been so far verified based on extensive numerical case studies as well as experimental investigation. Nevertheless, some practical considerations need to be addressed regarding the practical implementation of the proposed method. In practice, when mass modification is made at one node, all 6 degrees of freedom should be taken into account. So, for a unit mass, the modification matrix $[\mathbf{E}_{ii}]$ becomes,

$$[\mathbf{E}_{ii}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & 0 & 0 \\ 0 & 0 & 0 & 0 & I_y & 0 \\ 0 & 0 & 0 & 0 & 0 & I_z \end{bmatrix} \tag{37}$$

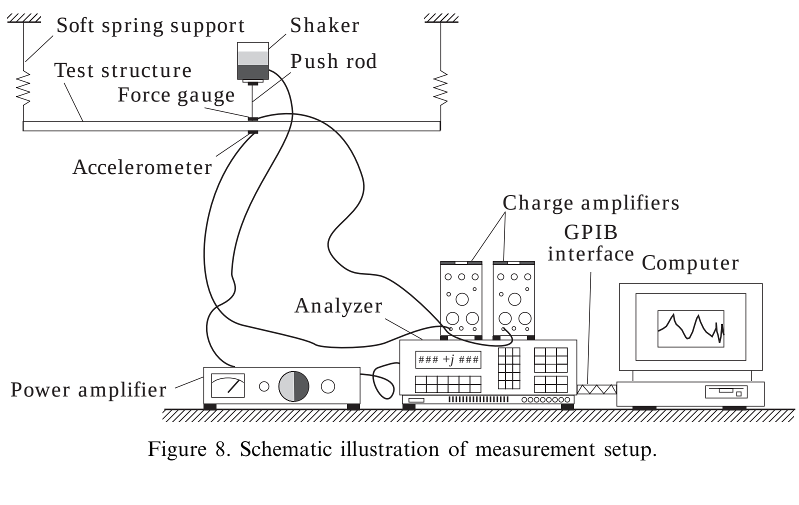

Figure 8. Schematic illustration of measurement setup.

where I_x, I_y and I_z are the second moments of inertia which can be determined when the geometry of the mass modification is known. However, rotation co-ordinates are very difficult to measure in practice [18]. Fortunately, the effect of the second moment of inertia is generally not significant in the case of mass modification, $[\mathbf{E}_{ii}]$ can be condensed to become,

$$[\mathbf{E}_{ii}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{38}$$

In the special case where only one translational motion needs to be considered such as beams and plates, $[\mathbf{E}_{ii}]$ can be further simplified to become that which was discussed in section 2. So, in the case of mass modification, provided translational degrees of freedom are measured, structural sensitivities with respect to mass changes can be accurately

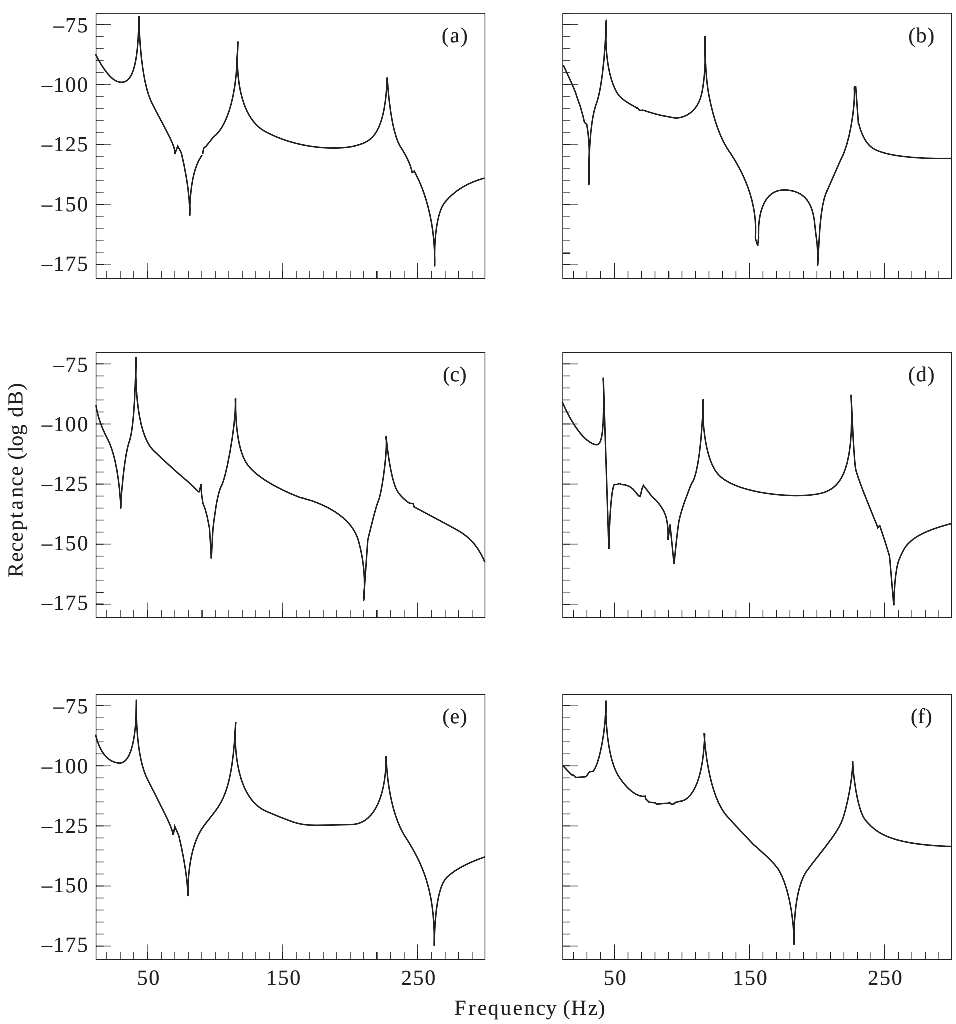

Figure 9. Measured frequency response functions: (a) measured receptance $\alpha_{13}(\omega)$; (b) measured receptance $\alpha_{23}(\omega)$; (c) measured receptance $\alpha_{33}(\omega)$; (d) measured receptance $\alpha_{43}(\omega)$; (e) measured receptance $\alpha_{53}(\omega)$; (f) measured receptance $\alpha_{63}(\omega)$.

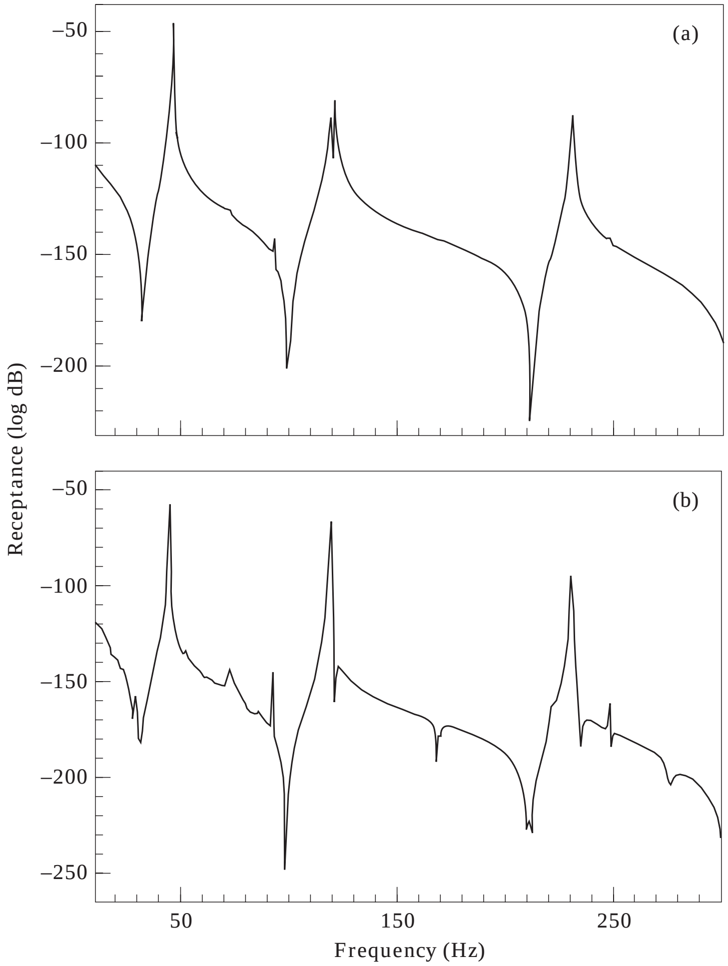

Figure 10. Receptance sensitivities of $\partial\alpha_{33}(\omega)/\partial m_3$; (a) real part of $\partial\alpha_{33}(\omega)/\partial m_3$; (b) imaginary part of $\partial\alpha_{33}(\omega)/\partial m_3$.

determined based on the proposed method. In the case of stiffness modification, the modifications are often made in the form of stiffeners. The important step here is to determine the modification matrix $[\mathbf{E}_{ij}]$ when rotational co-ordinates are not measured. This can be done by establishing the element stiffness matrix of the added stiffener first

TABLE 5

Eigenvector sensitivities of the test beam structure

Co-ord. No.	Mode 1		Mode 2		Mode 3	
	$\mathrm{Re}(\partial\{\boldsymbol{\phi}\}_1/\partial m_3)$	$\mathrm{Im}(\partial\{\boldsymbol{\phi}\}_1/\partial m_3)$	$\mathrm{Re}(\partial\{\boldsymbol{\phi}\}_2/\partial m_3)$	$\mathrm{Im}(\partial\{\boldsymbol{\phi}\}_2/\partial m_3)$	$\mathrm{Re}(\partial\{\boldsymbol{\phi}\}_3/\partial m_3)$	$\mathrm{Im}(\partial\{\boldsymbol{\phi}\}_3/\partial m_3)$
x_1	0·78796	0·10527	−2·38918	−0·08621	−2·64051	0·12901
x_2	1·22051	−0·09862	1·92124	0·11902	1·86763	−0·05873
x_3	1·50423	−0·13654	2·31390	−0·13982	1·57124	0·11762
x_4	1·61001	0·09672	1·65142	−0·06892	1·48253	−0·08962
x_5	0·67887	−0·04561	0·13975	0·04481	−1·35315	−0·14021
x_6	−0·97411	0·10881	−1·18073	0·06742	1·99709	0·15082

by methods such as finite element analysis and then condense the element stiffness matrix by retaining only those measured translational co-ordinates using reduction techniques such as a Guyan reduction [24]. From the condensed element stiffness matrix, modification matrix $[\mathbf{E}_{ij}]$ can be established in a similar way as in the case of mass modification.

5. CONCLUDING REMARKS

Many different methods have been developed for the efficient computations of structural sensitivities which are often required in structural dynamic analyses. Though these existing methods have been proven to be very useful tools to structural analysts, they are restricted to those cases where accurate analytical or finite element models are available. In many practical applications, where only limited measured data are available, while sensitivities are needed in the solution of troubleshooting problems such as modifications of existing structural systems, condition monitoring and failure detection, existing methods are inapplicable and the development of some alternative methods becomes necessary. In the present paper, a new and effective method has been developed to calculate the receptance, eigenvalue and eigenvector sensitivities from limited vibration test data. Such sensitivities are more accurate than those calculated from analytical or finite element models since structural modelling errors are inevitable due to the complexity of engineering structures. Extensive numerical case studies as well as systematic experimental investigations have been conducted and results have demonstrated the practicality of the proposed method.

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